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IRAM-30m EMIR time/sensitivity estimator

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Abstract

This memo describes the equations used in the IRAM-30m EMIR time/sensitivity estimator available in the `GILDAS/ASTRO` program. A large part of the memo aims at deriving sensitivity estimate for the case of On-The-Fly observations, which is not clearly documented elsewhere (to our knowledge). Numerical values of the different parameters used in the time/sensitivity estimator are grouped in appendix A.

Contents

1 Generalities

1.1 The radiometer equation

The radiometer equation for a total power measurement reads

$$\sigma = \frac{T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\, t}}, \tag{1}$$

where σ is the rms noise obtained by integration during t in a frequency resolution $d\nu$ with a system whose system temperature is given by $T_{\rm sys}$ and spectrometer efficiency is $\eta_{\rm spec}$. However, total power measurement includes other contributions (*e.g.* the atmosphere emission) in addition to the astronomical signal. The usual way to remove most of the unwanted contributions is to switch, *i.e.* to measure alternatively on-source and off-source and then to subtract the off-source from the on-source measurements. It is easy to show that the rms noise of the obtained measurement is

$$\sigma = \sqrt{\sigma_{\rm on}^2 + \sigma_{\rm off}^2} = \frac{T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\, t_{\rm sig}}} \quad \text{with} \quad t_{\rm sig} = \frac{t_{\rm on}\, t_{\rm off}}{t_{\rm on} + t_{\rm off}}, \tag{2}$$

where $\sigma_{\rm on}$ and $\sigma_{\rm off}$ are the noise of the on and off measurement observed respectively during the $t_{\rm on}$ and $t_{\rm off}$ integration time. $t_{\rm sig}$ is just a useful intermediate quantity.

1.2 System temperature

The system temperature is a summary of the noise added by the system. This noise comes from 1) the receiver and the optics, 2) the emission of the sky, and 3) the emission picked up by the secondary side lobes of the telescope. It is usual to approximate it (in the $T_{\rm a}^\star$ scale) with

$$T_{\rm sys} = \frac{(1+G_{\rm im})\exp\{\tau_{\rm s}\, A\}}{F_{\rm eff}} \left[F_{\rm eff}\, T_{\rm atm}\, \left(1-\exp\{-\tau_{\rm s}\, A\}\right) + (1-F_{\rm eff})\, T_{\rm cab} + T_{\rm rec}\right], \tag{3}$$

where $G_{\rm im}$ is the receiver image gain, $F_{\rm eff}$ the telescope forward efficiency, $A = 1/\sin(\text{elevation})$ the airmass, $\tau_{\rm s}$ the atmospheric opacity in the signal band, $T_{\rm atm}$ the mean physical atmospheric temperature, $T_{\rm cab}$ the ambient temperature in the receiver cabine and $T_{\rm rec}$ the noise equivalent temperature of the receiver and the optics. All those parameters are easily measured, except $\tau_{\rm s}$, which is depends on the amount of water vapor in the atmosphere and which is estimated by complex atmospheric models.

1.3 Elapsed telescope time

The goal of a time estimator is to find the elapsed telescope time ($t_{\rm tel}$) needed to obtain a given rms noise, while a sensitivity estimator aims at finding the rms noise obtained when observing during $t_{\rm tel}$. If $t_{\rm onoff}$ is the total integration time spent both on the on-source and off-source observations, then

$$t_{\rm onoff} = \eta_{\rm tel}\, t_{\rm tel}, \tag{4}$$

where $\eta_{\rm tel}$ is the efficiency of the observing mode, *i.e.* the time needed 1) to do calibrations (*e.g.* pointing, focus, temperature scale calibration), and 2) to slew the telescope between useful integrations.

The tuning of the receivers is not proportional to the total integration time but it should be added to the elapsed telescope time. A time estimator can hardly anticipate the total tuning time for a project. Indeed, one project (*e.g.* faint line detection) can request only one tuning to be used during many hours and another (*e.g.* line survey) can request a tuning every few minutes. In our case, we thus request that the estimator user add by hand the tuning time to the elapsed telescope time estimation.

1.4 Switching modes and observation kinds

Switching is done in two main ways.

Position switch where the off-measurement is done on a close-by sky position devoid of signal. Wobbler switching is a particular case.

Frequency switch where the telescope always points towards the the source and the switching is done in the frequency (velocity) space.

Moreover, there are two main observation kinds.

Tracked observations where the telescope track the source, *i.e.* it always observes the same position in the source referential. The result is a single spectra.

On-The-Fly observations where the telescope continuously slew through the source with time to map it. The result is a cube of spectra.

In the following, we will work out the equations needed by the time/sensitivity estimator for each combination.

2 Tracked observations

2.1 Frequency switched

In this case, all the time is spent in the direction of the source. However, the frequency switching also implies that all this times can be counted as on-source and off-source times. Thus

$$t_{\rm onoff} = t_{\rm on} = t_{\rm off}, \tag{5}$$

$$t_{\rm sig} = \frac{t_{\rm on}}{2} = \frac{t_{\rm off}}{2} = \frac{t_{\rm onoff}}{2}, \tag{6}$$

and

$$\sigma_{\rm fsw} = \frac{\sqrt{2}\,T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\,\eta_{\rm tel}\,t_{\rm tel}}}. \tag{7}$$

2.2 Position switched

In this case, only half of the time is spent in the direction of the source. Thus

$$t_{\rm on} = t_{\rm off} = \frac{t_{\rm onoff}}{2}, \tag{8}$$

$$t_{\rm sig} = \frac{t_{\rm on}}{2} = \frac{t_{\rm off}}{2} = \frac{t_{\rm onoff}}{4}, \tag{9}$$

and

$$\sigma_{\rm fsw} = \frac{2\,T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\,\eta_{\rm tel}\,t_{\rm tel}}}. \tag{10}$$

2.3 Comparison

For tracked observations, position switched observations results in a noise rms $\sqrt{2}$ larger than frequency switched observations for the same elapsed telescope time. In other words, frequency switched observations are twice as efficient in time to reach the same rms noise than position switched observations.

However, time efficiency is not the only criteria of choice. Indeed, with the current generation of receivers (before march 2009), the IF bandpass is much cleaner in position switched than in frequency switched observations. Frequency switched is thus really useful only when the lines are narrow so that the IF bandpass can be easily cleaned out through baselining with low order polynomials.

3 On-The-Fly observations

3.1 Additional notions and notations

The On-The-Fly (OTF) observing mode is used to map a given region of the sky. The time/sensitivity estimator will have to link the elapsed telescope time to cover the whole mapped region to the sensitivity in each independent resolution element. To do this, we need to introduce

- $A_{\rm map}$ and $A_{\rm beam}$, which are respectively the area of the map and the area of the resolution element. The map area is a user input while the resolution area is linked to the telescope full width at half maximum (θ) by

$$A_{\rm beam} = \frac{\eta_{\rm grid}\,\pi\,\theta^2}{4} \tag{11}$$

  where $\eta_{\rm grid}$ comes from the fact that the OTF data is gridded by convolution. When the convolution kernel is a Gaussian of FWHM equal to $\theta/3$ (the default inside the `GILDAS/CLASS` software), it is easy to show that

$$\eta_{\rm grid} = 1 + \frac{1}{9} \simeq 1.11. \tag{12}$$

- The number of independent measurement ($n_{\rm beam}$) in the final map which is given by

$$n_{\rm beam} = \frac{A_{\rm map}}{A_{\rm beam}}. \tag{13}$$

- The on and off time spent per independent measurement, $t_{\rm on}^{\rm beam}$ and $t_{\rm off}^{\rm beam}$. $t_{\rm sig}$ can then be written

$$t_{\rm sig} = \frac{t_{\rm on}^{\rm beam}\,t_{\rm off}^{\rm beam}}{t_{\rm on}^{\rm beam} + t_{\rm off}^{\rm beam}} \tag{14}$$

- The on and off time spent to map the whole map, $t_{\rm on}^{\rm tot}$ and $t_{\rm off}^{\rm tot}$. $t_{\rm onoff}$ is deduced from $t_{\rm on}^{\rm tot}$ and $t_{\rm off}^{\rm tot}$ in a way which depends on the switching scheme.

In addition, we must ensure that the user does not try to scan faster than the telescope can slew. To do this, we need to introduce

- The linear scanning speed, $v_{\rm linear}$, and its maximum value, $v_{\rm linear}^{\rm max}$.
- The area scanning speed, $v_{\rm area}$, and its maximum value, $v_{\rm area}^{\rm max}$. When the scanning pattern is linear, then $v_{\rm area}$ and $v_{\rm linear}$ are linked through

$$v_{\rm area} = v_{\rm linear}\,\Delta\theta, \tag{15}$$

  where $\Delta\theta$ is the separation between consecutive rows. To avoid nasty signal and noise aliasing problems, we must ensure a Nyquist sampling, *i.e.*

$$\Delta\theta = \frac{\theta}{2}. \tag{16}$$

3.2 Frequency switched

In frequency switched observations, the switching happens as the telescope is slewed. This is correct as long as the switching time is much smaller than the time needed to slew a significant fraction of the telescope beam.

It is easy to understand that

$$t_{\rm onoff} = t_{\rm on}^{\rm tot} = t_{\rm off}^{\rm tot}, \tag{17}$$

$$t_{\rm on}^{\rm beam} = t_{\rm off}^{\rm beam} = \frac{t_{\rm onoff}}{n_{\rm beam}}, \tag{18}$$

$$t_{\rm sig} = \frac{t_{\rm on}^{\rm beam}}{2} = \frac{t_{\rm off}^{\rm beam}}{2} = \frac{t_{\rm onoff}}{2n_{\rm beam}}, \tag{19}$$

and

$$\sigma_{\rm fsw} = \frac{\sqrt{2\,n_{\rm beam}}\,T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\,\eta_{\rm tel}\,t_{\rm tel}}}. \tag{20}$$

The velocity check can then be written as

$$\frac{A_{\rm map}}{t_{\rm onoff}} \le v_{\rm area}^{\rm max}. \tag{21}$$

3.3 Position switched

When the stability of the system is long enough, we can share the same off for several *independent* on-positions measured in a row (*e.g.* ON-ON-ON-OFF-ON-ON-ON-OFF...). The first key point here is the fact that the on-positions must be independent. The OTF is an observing mode where the sharing of the off can be used because the goal is to map a given region of the sky made of independent positions or resolution elements. When sharing the off-position between several on, Ball (1976) showed that the optimal off integration time is

$$t_{\rm off}^{\rm optimal} = \sqrt{n_{\rm on/off}}\,t_{\rm on} \tag{22}$$

where $n_{\rm on/off}$ is the number of on measurements per off. Replacing $t_{\rm off}$ by its optimal value in eq. 2, we obtain

$$t_{\rm sig} = \frac{t_{\rm on}}{1+\frac{1}{\sqrt{n_{\rm on/off}}}} \quad \text{and} \quad \sigma = \frac{T_{\rm sys}}{\eta_{\rm spec}\sqrt{d\nu\,t_{\rm on}}}\sqrt{1+\frac{1}{\sqrt{n_{\rm on/off}}}}. \tag{23}$$

We thus see that the rms noise decreases when the number of independent on per off increases. It seems tempting to have only one off for all the on positions of the OTF map. However, the second key point of the method is that the system must be stable between the first and last on measurement. To take this point into account we must introduce

- The concept of submap, which is a part of a map observed between two successive off measurements.
- $A_{\rm submap}$, which is the area covered by the telescope in each submap.
- $n_{\rm submap}$ the number of such submaps needed to cover the whole map area.
- $t_{\rm stable}$, the typical time where the system is stable. This time will be the maximum time between two off measurements, which is noted $t_{\rm submap}$.
- $n_{\rm cover}$, the number of coverages needed either to reach a given sensitivity or to exhaust the acquisition time.

By construction

- The submap area is the product of the area velocity by the time to cover it

$$A_{\rm submap} = v_{\rm area}\,t_{\rm submap}. \tag{24}$$

- The number of on per off is the number independent resolution elements in each submap

$$n_{\rm on/off} = \frac{A_{\rm submap}}{A_{\rm beam}}. \tag{25}$$

- The number of submap is the are of the map divided by the area of a submap

$$n_{\rm submap} = \frac{A_{\rm map}}{A_{\rm submap}}. \tag{26}$$

- The number of independent resolution elements in the map is the product of number of submaps by the number of on per off

$$n_{\rm beam} = n_{\rm submap}\, n_{\rm on/off}. \tag{27}$$

- The total on+off time is the product of the time needed to cover the whole map area once (which is the product of $t_{\rm submap}$ by $n_{\rm submap}$) by the number of map coverage needed

$$t_{\rm onoff} = n_{\rm cover}\, n_{\rm submap}\, t_{\rm submap}. \tag{28}$$

- The time to scan a submap is the sum of the $n_{\rm on/off}$ independent on integration time plus the associated off integration time

$$t_{\rm submap} = n_{\rm on/off}\, t_{\rm on} + t_{\rm off} = \left(n_{\rm on/off} + \sqrt{n_{\rm on/off}}\right) t_{\rm on}, \tag{29}$$

where $t_{\rm on}$ and $t_{\rm off}$ are the times spent on and off per independent measurement per coverage. $t_{\rm on}^{\rm beam}$ and $t_{\rm off}^{\rm beam}$ are just deduced by multiplication by the number of coverages

$$t_{\rm on}^{\rm beam} = \frac{n_{\rm cover}\, t_{\rm submap}}{n_{\rm on/off} + \sqrt{n_{\rm on/off}}} \quad \text{and} \quad t_{\rm off}^{\rm beam} = \frac{n_{\rm cover}\, \sqrt{n_{\rm on/off}}\, t_{\rm submap}}{n_{\rm on/off} + \sqrt{n_{\rm on/off}}}. \tag{30}$$

Using eqs. 29 and 30 to remplace $t_{\rm on}$ in eq. 23, we obtain

$$t_{\rm sig} = \frac{n_{\rm cover}\, t_{\rm submap}}{\left(1 + \sqrt{n_{\rm on/off}}\right)^2}. \tag{31}$$

Once these equations are written, we are in theory able to find the rms noise as a function of the elasped telescope time (sensitivity estimation) and vice-versa (time estimation). However, it is not fully straightforward because we must enforce that $n_{\rm cover}$ and $n_{\rm submap}$ have an integer value. To do this, we can play on the values of $t_{\rm submap}$ and $v_{\rm area}$ with the condition that they must be at maximum respectively $t_{\rm stable}$ and $v_{\rm area}^{\rm max}$. This additional step depends on the kind of estimation (either a sensitivity or a time estimation).

3.3.1 Sensitivity estimation

Computation of $n_{\rm cover}$ and $v_{\rm area}$ Using Eqs. 24, 26 and 28, we obtain

$$n_{\rm cover} = \frac{v_{\rm area}\, t_{\rm onoff}}{A_{\rm map}}. \tag{32}$$

Using this equation, we start to compute $n_{\rm cover}$ for $v_{\rm area} = v_{\rm area}^{\rm max}$. There are then two cases.

1. $n_{\rm cover} < 1$: This means that the user tries to cover a too large sky area in the given telescope elasped time.
2. $n_{\rm cover} \geq 1$: We then enforce the integer character of $n_{\rm cover}$ with

$$n_{\rm cover} = \texttt{int}(n_{\rm cover}), \tag{33}$$

and we recompute the associated area velocity

$$v_{\rm area} = \frac{n_{\rm cover}\, A_{\rm map}}{t_{\rm onoff}}. \tag{34}$$

The use of the `int()` operator ensures that $v_{\rm area} < v_{\rm area}^{\rm max}$.

Computation of $n_{\rm submap}$ and $t_{\rm submap}$ Using Eqs. 24 and 26, we obtain

$$n_{\rm submap} = \frac{A_{\rm map}}{v_{\rm area}\, t_{\rm submap}}. \tag{35}$$

Using this equation, we start to compute $n_{\rm submap}$ for $t_{\rm submap} = t_{\rm stable}$. We want to enforce the integer character of $n_{\rm submap}$ in a way which decreases $t_{\rm submap}$. To do this, we use

$$n_{\rm submap} = 1 + \texttt{int}(n_{\rm submap}), \tag{36}$$

and we recompute the associated submap time

$$t_{\rm submap} = \frac{A_{\rm map}}{v_{\rm area}\, n_{\rm submap}}. \tag{37}$$

Eq. 36 ensures that $t_{\rm submap} < t_{\rm stable}$.

3.3.2 Time estimation

Computation of $n_{\rm submap}$, $A_{\rm submap}$ We start by computing the maximum submap area with

$$A^{\rm max}_{\rm submap} = v^{\rm max}_{\rm area}\, t_{\rm submap} \quad \text{and} \quad t_{\rm submap} = t_{\rm stable}. \tag{38}$$

Then eq. 26 allows us to compute a value of $n_{\rm submap}$. We enforce the integer character of $n_{\rm submap}$ through

$$n_{\rm submap} = 1 + \texttt{int}(n_{\rm submap}). \tag{39}$$

We then compute the new value of $A_{\rm submap}$ with

$$A_{\rm submap} = \frac{A_{\rm map}}{n_{\rm submap}}. \tag{40}$$

Computation of $n_{\rm cover}$, $v_{\rm area}$ and $t_{\rm submap}$ Because we are doing a time estimation, the noise rms is a user input which allows us to compute $t_{\rm sig}$ through the eq. 2. Using eqs. 13, 24, 27 and 31, we obtain

$$n_{\rm cover} = t_{\rm sig}\, v_{\rm area} \left(\frac{1}{\sqrt{A_{\rm submap}}} + \frac{1}{\sqrt{A_{\rm beam}}} \right)^2. \tag{41}$$

We start to compute $n_{\rm cover}$ with $v_{\rm area} = v^{\rm max}_{\rm area}$. There are two cases.

1. $n_{\rm cover} < 1$. To enforce the integer character of $n_{\rm cover}$, we will have to set $n_{\rm cover} = 1$. Looking at the form of eq. 41, there is no other adjustment factor than $t_{\rm sig}$ (because $v_{\rm area}$ is already at its maximum value). In other words, we need a minimum time to cover $A_{\rm map}$ at the maximum velocity possible with the telescope and this minimum time implies a more sensitive observation than requested by the user.

2. $n_{\rm cover} \geq 1$. If $n_{\rm cover}$ is not an integer, we can think to decrease $v_{\rm area}$ from $v^{\rm max}_{\rm area}$ to obtain an integer value. However, this must be done at constant $A_{\rm submap} (= v_{\rm area}\, t_{\rm submap})$. Decreasing $v_{\rm area}$ thus implies increasing $t_{\rm submap}$. It is not clear that this is possible because of the constraint $t_{\rm submap} < t_{\rm stable}$. Another way to deal with this is to keep $v_{\rm area}$ to its maximum value and to adjust $t_{\rm sig}$ and thus σ to obtain an integer value of $n_{\rm cover}$. This means that the resulting sensitivity would be either less (case: $n_{\rm cover} = \texttt{int}(n_{\rm cover})$) or more (case: $n_{\rm cover} = \texttt{int}(n_{\rm cover}) + 1$) than the requested one. The worst case is when $n_{\rm cover}$ is changing from 1 to 2 because it can double the elapsed telescope time. The larger the value of $n_{\rm cover}$ the less harm it is to enforce the integer character of $n_{\rm cover}$.

3.4 Comparison

Contrary to tracked observations, the position switched observing mode can be more efficient than the frequency switched observing mode. Indeed, in frequency switch, the same time is spent in the on and off spectrum. When subtracting them, the off brings as much noise as the on. In position switch, the same off can be shared between many ons, in which case the optimal integration time on the off is much larger than on each independent on spectrum. Hence, the noise brought by the off spectrum can be much smaller than the noise brought by the on spectrum.

For frequency switched observations,

$$t_{\mathrm{sig}} = \frac{t_{\mathrm{onoff}}}{2\, n_{\mathrm{beam}}}, \tag{42}$$

while for position switched observations, we can derive from eqs. 27, 31 and 30

$$t_{\mathrm{sig}} = \frac{t_{\mathrm{onoff}}}{n_{\mathrm{beam}} \left(1 + \frac{1}{\sqrt{n_{\mathrm{on/off}}}}\right)^2}. \tag{43}$$

We see that position switched OTF starts to be more efficient than frequency switched OTF for

$$n_{\mathrm{on/off}} = \frac{1}{3 - 2\sqrt{2}} \sim 6. \tag{44}$$

Moreover, $\left(1 + \frac{1}{\sqrt{n_{\mathrm{on/off}}}}\right)^2 \simeq 1.4$ for $n_{\mathrm{on/off}} = 30$, and $\simeq 1.2$ for $n_{\mathrm{on/off}} = 100$. Using eqs. 24 and 25, we see that the limit on the maximum number of on per off is set by

$$n_{\mathrm{on/off}} = \frac{t_{\mathrm{stable}}}{A_{\mathrm{beam}}/v_{\mathrm{area}}^{\mathrm{max}}}, \tag{45}$$

i.e. the ratio of the maximum system stability time by the minimum time required to map a telescope beam.

As for tracked observations, there are other considerations to be taken into account. For extra-galactic observations, the lines are large which excludes the use of frequency switched observations. For Galactic observations, the intrinsic sensitivity of the receivers may make it difficult to find a closeby position devoid of signal. We can still use the position switched OTF observing mode. But we then have to observe the off position in frequency switched track observing mode long enough to be able to add back the off astronomical signal. Here I should do the computation of the needed time.

4 Acknowledgement

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References

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A Numerical values

This appendix groups all the numerical values used in the time/sensitivity estimator. We made conservative choices for two reasons: 1) time/sensitivity estimators tend to be too optimistics and 2) EMIR is a new generation of receivers which had not yet been tested at the telescope.

A.1 Overheads

- $\eta_{\rm tel} = 0.5$.
- After estimating the number of tunings needed to complete the project, the user has to add to the telescope time 30 minutes per tuning (this includes the observation of a line calibrator).

A.2 Atmosphere

- $T_{\rm atm} = 250$ K.
- The opacities at signal frequencies are computed with a recent version of the ATM program (maintained by J.R. Pardo).
- They are computed for 3 different amount of water vapor per season (1, 2 and 4 mm for the winter season and 2, 4 and 7 mm for the summer season).

A.3 Telescope

- $T_{\rm cab} = 290$ K.
- $F_{\rm eff} = 0.95$ at 3 mm, 0.93 at 2 mm, 0.91 at 1 mm and 0.88 at 0.8 mm.

A.4 Frontends

Warning: Please do not quote these values in your papers. You should refer to the publications which fully describe the receivers.

- The receiver temperature is the sum of
  - The mixer temperature: Typically 50 K below 260 GHz and 70 K above;
  - The mirror losses: Typically 10 K;
  - The dichroic losses: Typically 15 K. Nota Bene: Dichroics enable dual frequency observation by frequency separation of the sky signal.

  We end up with $T_{\rm rec} = 75$ K below 260 GHz and $Trec = 95$ K above 260 GHz.
- $G_{\rm im} = 0.1$.

A.5 Backends

- $\eta_{\rm spec} = 0.87$ because of the 2 bit quantization at the input of the correlators.
- The noise equivalent bandwidth of our correlators is almost equal to the channel spacing. So we do not take this into account in our estimation.

A.6 On-The-Fly

- $t_{\mathrm{stable}} = 5$ minutes.
- $\theta = \frac{2460''}{\nu\,\mathrm{GHz}}$.
- The maximum linear velocity is limited by the maximum dumping rate of 2 Hz. We know that in order to avoid beam elongation along the scanning direction, we need to sample at least 4 points per beam in the scanning direction. We thus end up with

  $$v_{\mathrm{linear}}^{\mathrm{max}} = 0.5\,\theta\,\mathrm{arcsec/s} \tag{46}$$

  and

  $$v_{\mathrm{area}}^{\mathrm{max}} = 0.25\,\theta^2\,\mathrm{arcsec}^2/\mathrm{s}. \tag{47}$$

B Optimal number of ON per OFF measurements

This section is just a reformulation of the original demonstration by Ball (1976).

Let's assume that we are measuring $n_{\mathrm{on/off}}$ *independent* on-positions for a single off. The same integration time (t_{on}) is spent on each on-position and the off integration time is

$$t_{\mathrm{off}} = \alpha\, t_{\mathrm{on}}, \tag{48}$$

where α can be varied. Using eq. 2 and $t_{\mathrm{onoff}} = n_{\mathrm{on/off}} t_{\mathrm{on}} + t_{\mathrm{off}} = (n_{\mathrm{on/off}} + \alpha)\, t_{\mathrm{on}}$, it can be shown than

$$t_{\mathrm{onoff}} = \frac{T_{\mathrm{sys}}^2}{\eta_{\mathrm{spec}}^2\, \sigma^2\, d\nu} \left(1 + n_{\mathrm{on/off}} + \alpha + \frac{n_{\mathrm{on/off}}}{\alpha}\right). \tag{49}$$

Differenciating with respect to α, we obtain

$$\frac{dt_{\mathrm{onoff}}}{d\alpha} \propto 1 - \frac{n_{\mathrm{on/off}}}{\alpha^2} \tag{50}$$

Setting the result to zero then gives that the minimum elapsed time to reach a given rms noise is obtained for

$$\alpha = \sqrt{n_{\mathrm{on/off}}} \quad \text{or} \quad t_{\mathrm{off}}^{\mathrm{optimal}} = \sqrt{n_{\mathrm{on/off}}}\, t_{\mathrm{on}}. \tag{51}$$